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Abstract

We consider in this article a model of vesicle moving into a viscous incompressible fluid. The vesicle is described through a phase-field equation and through a transport equation modeling the local incompressibility of its membrane. The equations for the fluid are the classical Navier–Stokes equations with a force resulting from the presence of the vesicle. The model for the system vesicle–fluid is an approximation of a model obtained by Jamet and Misbah [10]. Our main result states the existence of weak solutions for the corresponding system. The proof is based on compactness/monotonicity arguments

1 Introduction and main result

This article is devoted to the study of model of a vesicle moving into a viscous incompressible fluid. A vesicle is a closed membrane which usually contains and is contained into a fluid. It has a simple structure and thus may be seen as a first step to understand more complicated entities such as red blood cells. Indeed, they can exhibit many interesting behaviors such as "tank treading", "tumbling" or "vacillating-breathing". They are also used to study blood rheology.

One could model the membrane position as a closed surface. In that case, one has to deal with two fluids separated by a sharp interface which is moving and which evolution depends on the two fluids. Such a free boundary problem can be very difficult to handle both from theoretical and numerical points of view. Another model which is more interesting from numerical aspects consists in using a phase-field function and to replace the sharp interface by a diffuse interface. More precisely, one consider a real function ϕ defined in the whole spatial domain Ω and which rapid variation corresponds to the domain of the vesicle. One can then assume that there is only one fluid which fills also the whole domain Ω and which physical parameters could depend on ϕ. One of the difficulties consists in obtaining equations for ϕ. One way to do this is to use phase-field models which are based on energy functional. The model we consider is a phase-field model derived by Jamet and Misbah [10] where they include a thermodynamical approach. In their model, they take into account a property of vesicle which is the local incompressibility of the membrane. In order to do that, they introduce a tension–field ζ and in the energy of the system they add to the "usual" phase-field terms, a term related to the local incompressibility and a corresponding term which comes from thermodynamical aspects. After a change of variable, they replace ζ by a

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quantity σ defined through ζ and ϕ and which is only convected by the fluid. Their system could be written as

$$\begin{cases} u_t + u\cdot\nabla u = \Delta u + \nabla p - \nabla\cdot\left(\left(\lambda + \dfrac{\sigma}{|\nabla\phi|}\right)\nabla\phi\otimes\nabla\phi\right), \\ \nabla\cdot u = 0, \\ \phi_t + u\cdot\nabla\phi - \nabla\cdot\left(\left(\lambda+\dfrac{\sigma}{|\nabla\phi|}\right)\nabla\phi\right) = -W'(\phi), \\ \sigma_t + u\cdot\nabla\sigma = 0. \end{cases} \tag{1.1}$$

In the above system, we have simplified the system of [10] by taking many physical quantities constant and equal to 1 (density, viscosity, etc.). Moreover, we have slightly modified some notation: in [10], the phase-field equation in (1.1) is written as

$$\phi_t + u\cdot\nabla\phi - \nabla\cdot\left(\left([\lambda+\theta] - \frac{[\theta\sigma]}{|\nabla\phi|}\right)\nabla\phi\right) = -W'(\phi),$$

for some constant θ. The last term in the first equation of (1.1) is also written as

$$-\nabla\cdot\left(\left([\lambda+\theta] + \frac{[\theta\sigma]}{|\nabla\phi|}\right)\nabla\phi\otimes\nabla\phi\right).$$

In this paper, we have replaced $[\lambda+\theta]$ by λ and $[\theta\sigma]$ by σ. Finally, it is worth noting that more complete system can be considered: for instance, one can add in the model the influence of the curvature of the surface (see [11]).

In the above system and in what follows, we write by $f_t = \frac{\partial f}{\partial t}$ the time derivative of a (vector or scalar) function. The other notation used above are detailed in Subsection 2.1 and in particular in (2.1).

From a mathematical point of view, the term $\frac{\nabla\phi}{|\nabla\phi|}$ may be quite difficult to handle and therefore, in this paper, we consider an approximation of (1.1). More precisely, assume $\epsilon > 0$ is a fixed real number. Then we consider the following system written in the domain $(0,T)\times\Omega$:

$$\begin{cases} u_t + u\cdot\nabla u = \Delta u + \nabla p - \nabla\cdot\left(\left(\lambda + \dfrac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}}\right)\nabla\phi\otimes\nabla\phi\right), & \text{(1.2a)} \\ \nabla\cdot u = 0, & \text{(1.2b)} \\ \phi_t + u\cdot\nabla\phi - \nabla\cdot\left(\left(\lambda + \dfrac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}}\right)\nabla\phi\right) = -W'(\phi), & \text{(1.2c)} \\ \sigma_t + u\cdot\nabla\sigma = 0, & \text{(1.2d)} \end{cases}$$

with the following boundary and initial conditions

$$\begin{cases} u(t,x) = 0, \quad \phi(t,x) = 0, \quad \text{for } (t,x)\in(0,T)\times\partial\Omega, & \text{(1.3a)} \\ u(0,x) = u_0(x), \quad \phi(0,x) = \phi_0(x), \quad \sigma(0,x) = \sigma_0(x), \text{ for } x\in\Omega. & \text{(1.3b)} \end{cases}$$

We assume in what follows that Ω is a bounded, smooth domain in $\mathbb{R}^3$ and λ is a positive constant. The function W is a "double-well" function, it is function with only two distinct minima ϕ_0 and ϕ_1. The idea in the phase-field model is that ϕ should approximate the piecewise constant function equal to ϕ_0 in the fluid outside the vesicle and ϕ_1 in the fluid contained in the vesicle. One can consider many choices for the function W, here we assume that $\phi_0 = 0$ and $\phi_1 = 1$ and we take

$$W(\phi) = \phi^2(\phi-1)^2, \quad W'(\phi) = 4\phi^3 - 6\phi^2 + 2\phi. \tag{1.4}$$

We are now in position to state our main result:

Theorem 1.1. *Assume*

$$u_0 \in V^0(\Omega),\ \ \phi_0 \in H_0^1(\Omega),\ \ \sigma_0 \in L^\infty(\Omega). \tag{1.5}$$

and $\sigma_0 \geqslant 0$ *almost everywhere in* Ω. *Then the system* (1.2)–(1.3) *admits a weak solution.*

The precise definition of a weak solution for system (1.2)–(1.3) is given in Section 2 (see Definition 2.1). The space $V^0(\Omega)$ used in the above theorem is defined as the closure in $[L^2(\Omega)]^3$ of the set $\mathscr{V}(\Omega)$ of the smooth compactly supported and divergence free functions

Let us give some remarks on Theorem 1.1. First, noticing that, $\sigma \equiv 0$ is a solution of our system (for $\sigma_0 \equiv 0$), we see that (1.2) (or (1.1)) is a generalization of the system

$$\left\{ \begin{array}{l} u_t + u \cdot \nabla u = \Delta u + \nabla p - \lambda \nabla \cdot (\nabla \phi \otimes \nabla \phi)\,, \\ \nabla \cdot u = 0, \\ \phi_t + u \cdot \nabla \phi = \lambda \Delta \phi - W'(\phi). \end{array} \right. \tag{1.6}$$

It is worth mentioning that a similar system was considered by Lin (see [12]) to model the flow of nematic liquid crystals with varying director lengths, or with variable degree of orientation:

$$\left\{ \begin{array}{l} u_t + u \cdot \nabla u = \Delta u + \nabla p - \lambda \nabla \cdot (\nabla d \bigodot \nabla d)\,, \\ \nabla \cdot u = 0, \\ d_t + u \cdot \nabla d = \lambda \Delta d - W'(d), \end{array} \right. \tag{1.7}$$

where $d : (0,T) \times \Omega \to \mathbb{R}^3$ is the optical molecule direction and

$$\nabla d \bigodot \nabla d = \left\{ \sum_{j=1}^{3} \partial_{x_i} d_j \partial_{x_j} d_k \right\}_{1 \leqslant i,j \leqslant 3}. \tag{1.8}$$

In [13], the authors studied the well-posedness of (1.7) with Dirichlet boundary conditions. In that work, they obtain several energy inequalities which enable them to obtain the existence of global weak solutions by employing an improved Galerkin method. They also discuss the uniqueness, regularity and some stability properties of solutions. Moreover, they proved in [14] that if the domain and the initial-boundary condition in problem (1.7) are smooth enough, then there exists a suitable weak solution whose singular set has one-dimension Hausdorff measure zero in space-time. Since system (1.7) contains the Navier-Stokes equations as a subsystem, this result can be considered as a natural generalization of an earlier work of Caffarelli-Kohn-Nirenberg (see [4]). We also refer at the works of [1] and [2] where similar problems are considered.

For general initial conditions σ_0, the method used in [13] seems difficult to adapt to our system (1.2)–(1.3). The main reason for this is that here we need to deal with the transport equation (1.2d) which solution appears in the coefficient of the operator of the phase-field equation (1.2c). In order to handle this problem and to keep uniform energy estimates, we choose another approximation which follows an idea presented in [15, p. 97] and which consists in adding some viscosity in the Navier–Stokes system so that the velocity of the approximate system is regular enough.

The plan of this paper is as follows. In Section 2, we derive a priori estimates for the system (1.2) and introduce the definition of weak solution. In Section 3, we obtain some preliminary inequalities and recall several classical results. Section 4 is devoted to the study of a viscous approximation of the Navier-Stokes equations whereas in Section 5 we show some results of well-posedness for the phase-field equation. Combining the results of these two sections, we can, in Section 6, solve the viscous approximation of (1.2)–(1.3) and the last section, Section 7, consists in passing to the limit as the artificial viscosity goes to 0 and thus prove the main result.

2 Notation, a priori estimates and definition of weak solution

In this section, we derive some a priori estimates and we introduce the definition of the solution for the system (1.2)–(1.3).

2.1 Notation

For two vector fields $u,\ v$, a scalar field ϕ and a second order tensor field $M=(m_{ij})$ on $\mathbb{R}^3$, we use the standard notation

$$u\otimes v=\{u_iv_j\}_{1\leqslant i,j\leqslant 3},\quad (\nabla\cdot M)_i:=\sum_j\frac{\partial m_{ij}}{\partial x_j},\quad (\nabla v)_{ij}:=\frac{\partial v_i}{\partial x_j},\quad \nabla^2\phi=\left(\frac{\partial\phi}{\partial x_i\partial x_j}\right)_{1\leqslant i,j\leqslant 3}. \tag{2.1}$$

Using the Leibniz rule, we have the following relation:

$$\nabla\cdot(u\otimes v)=(\nabla u)\,v+(\nabla\cdot v)\,u. \tag{2.2}$$

We also introduce here two important operators A_σ and J_σ:

$$A_\sigma(\phi):=-\nabla\cdot\left((\lambda+\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\right), \tag{2.3}$$

and

$$J_\sigma(\phi):=\left(\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}}\right)\nabla\phi. \tag{2.4}$$

In this work, we adopt standard notations of divergence-free vector fields of Sobolev spaces (see, for instance, [5, pp. 1-10]). Let us denote by $\mathscr{V}(\Omega)$ the set

$$\mathscr{V}(\Omega)=\{v\in(C_0^\infty(\Omega))^3\mid\nabla\cdot v=0\}$$

and by $V^k(\Omega)$ the closure of $\mathscr{V}(\Omega)$ in $H^k(\Omega)$ for $k\in\mathbb{N}^*$ and by $V^0(\Omega)$ the closure of $\mathscr{V}(\Omega)$ in $L^2(\Omega)$. It can be verified that for $k\geqslant 1$,

$$V^k(\Omega)=H_0^k(\Omega)\cap V^0(\Omega). \tag{2.5}$$

In particular, with this notation, the classical spaces $H(\Omega)$ and $V(\Omega)$ are denoted respectively by $V^0(\Omega)$ and $V^1(\Omega)$. We also denote by $V^{-k}(\Omega)$ the dual space of $V^k(\Omega)$ with respect to $V^0(\Omega)$. Finally, we set $Q^t=(0,t)\times\Omega$ for $t>0$.

Many constants arise in the course of our work. The symbol C denotes a generic constant whose value may change from line to line. We reserve subscripts $(\epsilon,\lambda,etc.)$ for those constants to which repeated reference must be made. We denote by $(\cdot,\cdot)$ the inner product of $L^2(\Omega)$ and by $\langle\cdot,\cdot\rangle$ the dual product between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

2.2 A priori estimates

Here, we formally compute some a priori estimates for the problem (1.2)–(1.3). We first write these equations in a different way by simple calculation. By using (2.2), we obtain

$$\begin{aligned}\nabla\cdot&\left((\lambda+\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\otimes\nabla\phi\right)\\&=\nabla^2\phi\left((\lambda+\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\right)+\nabla\cdot\left((\lambda+\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\right)\nabla\phi.\end{aligned} \tag{2.6}$$

On the other hand, using Leibniz rule implies

$$\nabla\left(\frac{\lambda}{2}|\nabla\phi|^2+\sigma\sqrt{|\nabla\phi|^2+\epsilon}\right)=\nabla^2\phi\left((\lambda+\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\right)+\left(\sqrt{|\nabla\phi|^2+\epsilon}\right)\nabla\sigma.$$

Combining the above equation with (2.6) and using the definition (2.3) of A_σ yield

$$\nabla\cdot\left((\lambda+\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\otimes\nabla\phi\right)$$
$$=-A_\sigma(\phi)\nabla\phi+\nabla\left(\frac{\lambda}{2}|\nabla\phi|^2+\sigma\sqrt{|\nabla\phi|^2+\epsilon}\right)-\left(\sqrt{|\nabla\phi|^2+\epsilon}\right)\nabla\sigma. \quad (2.7)$$

Consequently, if we denote by $q=q_{\phi,\sigma}$ the quantity

$$q_{\phi,\sigma}=\frac{\lambda}{2}|\nabla\phi|^2+\sigma\sqrt{|\nabla\phi|^2+\epsilon}, \quad (2.8)$$

then we can rewrite (1.2) as

$$u_t+u\cdot\nabla u=\Delta u+\nabla(p+q)+A_\sigma(\phi)\nabla\phi+\left(\sqrt{|\nabla\phi|^2+\epsilon}\right)\nabla\sigma \quad \text{in} \quad (0,T)\times\Omega, \quad (2.9a)$$
$$\nabla\cdot u=0 \quad \text{in} \quad (0,T)\times\Omega, \quad (2.9b)$$
$$\phi_t+u\cdot\nabla\phi=-W'(\phi)-A_\sigma(\phi) \quad \text{in} \quad (0,T)\times\Omega, \quad (2.9c)$$
$$\sigma_t+u\cdot\nabla\sigma=0 \quad \text{in} \quad (0,T)\times\Omega. \quad (2.9d)$$

- We multiply (2.9a) by u and integrate by parts:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2\,\mathrm{dx}+\int_\Omega |\nabla u|^2\,\mathrm{dx} = \int_\Omega A_\sigma(\phi)\nabla\phi\cdot u\,\mathrm{dx}+\int_\Omega \nabla\sigma\cdot u\sqrt{|\nabla\phi|^2+\epsilon}\,\mathrm{dx}. \quad (2.10)$$

- We multiply (2.9c) by $\phi_t+u\cdot\nabla\phi$ and integrate by parts:

$$\int_\Omega |\phi_t+u\cdot\nabla\phi|^2\,\mathrm{dx}=-\frac{d}{dt}\int_\Omega\left(W(\phi)+\frac{\lambda}{2}|\nabla\phi|^2+\sigma\sqrt{|\nabla\phi|^2+\epsilon}\right)\mathrm{dx}$$
$$-\int_\Omega A_\sigma(\phi)\nabla\phi\cdot u\,\mathrm{dx}+\int_\Omega \sigma_t\sqrt{|\nabla\phi|^2+\epsilon}\,\mathrm{dx}. \quad (2.11)$$

Summing (2.10) and (2.11), we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\int_\Omega |u|^2+W(\phi)+\frac{\lambda}{2}|\nabla\phi|^2+\sigma\sqrt{|\nabla\phi|^2+\epsilon}\right)\mathrm{dx}+\int_\Omega |\nabla u|^2+\int_\Omega |\phi_t+u\cdot\nabla\phi|^2\,\mathrm{dx}$$
$$=\int_\Omega(\sigma_t+\nabla\sigma\cdot u)\sqrt{|\nabla\phi|^2+\epsilon}\,\mathrm{dx}.$$

The above equation, (2.9c) and (2.9d) formally imply the following energy inequality:

$$E(t)+\int_0^t\int_\Omega |\nabla u|^2+|W'(\phi)+A_\sigma(\phi)|^2\,\mathrm{dx}\,\mathrm{d}s\leqslant E(0), \quad (2.12)$$

where the energy associated to our system is defined by

$$E(t):=\int_\Omega\left(\frac{1}{2}|u(t)|^2+\frac{\lambda}{2}|\nabla\phi(t)|^2+W(\phi(t))+\sigma\sqrt{|\nabla\phi(t)|^2+\epsilon}\right)\mathrm{dx}. \quad (2.13)$$

2.3 Definition of Weak Solutions

We are now in position to state the definition of weak solution of (1.2)–(1.3).

Definition 2.1 (Definition of weak solution)**.** *The triple* (u, ϕ, σ) *is a weak solution of* (1.2)–(1.3) *if*

$$\begin{cases} u \in L^2(0,T;V^1(\Omega)) \cap L^\infty(0,T;V^0(\Omega)), \\ \phi \in L^\infty(0,T;H_0^1(\Omega)), \\ \sigma \in C([0,T];L^p(\Omega)), \ (\forall p \in [1,\infty)). \end{cases} \tag{2.14}$$

if (u, ϕ, σ) *satisfies the energy estimate* (2.12) *and if the following relations hold:*

$$\begin{aligned} -\int_\Omega u_0 \cdot v(0,x)\,\mathrm{dx} - \int_{Q^T} u \cdot v_t \,\mathrm{dx}\,\mathrm{dt} - \int_{Q^T} (u \otimes u - \nabla u) : \nabla v \,\mathrm{dx}\,\mathrm{dt} \\ = \int_{Q^T} \nabla v : \left(\left(\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2 + \epsilon}} \right) \nabla\phi \otimes \nabla\phi \right) \mathrm{dx}\,\mathrm{dt}, \\ \textit{for all} \quad v \in C^1([0,T];V^6(\Omega)),\ v(T,\cdot) = 0, \end{aligned} \tag{2.15}$$

$$\begin{aligned} -\int_\Omega \phi_0 \psi(0,x)\,\mathrm{dx} - \int_{Q^T} \phi\psi_t \,\mathrm{dx}\,\mathrm{dt} + \int_{Q^T} \nabla\psi \cdot \left(\left(\lambda + \frac{\sigma}{\sqrt{|\nabla\phi|^2 + \epsilon}} \right) \nabla\phi \right) \mathrm{dx}\,\mathrm{dt} \\ = \int_{Q^T} (\phi u \cdot \nabla\psi - W'(\phi)\psi) \,\mathrm{dx}\,\mathrm{dt}, \\ \textit{for all} \quad \psi \in C^1([0,T];V^3(\Omega)),\ \psi(T,\cdot) = 0, \end{aligned} \tag{2.16}$$

$$\begin{aligned} -\int_\Omega \sigma_0 \eta(0,x)\,\mathrm{dx} - \int_{Q^T} \sigma\eta_t \,\mathrm{dx}\,\mathrm{dt} = \int_{Q^T} \sigma u \cdot \nabla\eta \,\mathrm{dx}\,\mathrm{dt}, \\ \textit{for all} \quad \eta \in C^1([0,T];V^3(\Omega)),\ \eta(T,\cdot) = 0. \end{aligned} \tag{2.17}$$

Remark 2.2. *The above definition is obtained by multiplying formally* (1.2a)*,* (1.2c) *and* (1.2d) *by smooth functions and integrating by parts. In particular, a smooth solution of* (1.2)–(1.3) *is a weak solution in the above sense. Note that in* (2.15)*,* (2.16) *and* (2.17)*, the terms appearing in the integral on* Q^T *and involving* u, ϕ, σ *are in* $L^1(Q^T)$ *if* (u, ϕ, σ) *satisfies* (2.14)*. This comes from Sobolev embedding theorems and Hölder inequalities, as it will be seen in Subsection 3.1.*

3 Preliminaries

In this section we recall some classical results and derive some basic inequalities which are useful in our problem.

3.1 Some inequalities

Let us assume (u, ϕ, σ) satisfies the regularity assumptions of Definition 2.1, i.e.

$$\begin{cases} u \in L^2(0,T;V^1(\Omega)) \cap L^\infty(0,T;V^0(\Omega)), \\ \phi \in L^\infty(0,T;H_0^1(\Omega)), \\ \sigma \in C([0,T];L^p(\Omega)), \ (\forall p \in [1,\infty)). \end{cases} \tag{3.1}$$

Using Hölder's inequalities, we obtain

$$\left((\lambda+\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\right)\in L^\infty(0,T;L^2(\Omega)),\quad \left((\lambda+\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\otimes\nabla\phi\right)\in L^\infty(0,T;L^1(\Omega))$$

with

$$\left\|\left((\lambda+\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\right)\right\|_{L^\infty(0,T;L^2(\Omega))}\leqslant C\left(\|\phi\|_{L^\infty(0,T;H^1(\Omega))}+\|\sigma\|_{L^\infty(0,T;L^2(\Omega))}\right),\tag{3.2}$$

and

$$\left\|\left((\lambda+\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\otimes\nabla\phi\right)\right\|_{L^\infty(0,T;L^1(\Omega))}\leqslant C\left(\|\phi\|^2_{L^\infty(0,T;H^1(\Omega))}+\|\sigma\|^2_{L^\infty(0,T;L^2(\Omega))}\right).\tag{3.3}$$

Using the Sobolev embedding $H^1(\Omega)\subset L^6(\Omega)$ and (1.4) we deduce that if $\phi\in L^\infty(0,T;H^1_0(\Omega))$, then

$$W'(\phi)\in L^\infty(0,T;L^2(\Omega))$$

and

$$\|W'(\phi)\|_{L^\infty(0,T;L^2(\Omega))}\leqslant C\left(\|\phi\|^3_{L^\infty(0,T;H^1(\Omega))}+\|\phi\|_{L^\infty(0,T;H^1(\Omega))}\right).\tag{3.4}$$

Using (3.3) and Sobolev embeddings, we can deduce the following proposition which gives some estimates of the nonlinear terms appearing in the right hand side of (1.2a). We skip the proof since its proof is classical.

Proposition 3.1. *Let $\sigma\in C([0,T];L^2(\Omega))$, $\phi\in L^4(0,T;H^1_0(\Omega))$ and*

$$F(\phi,\sigma)=-\nabla\cdot\left((\lambda+\frac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\otimes\nabla\phi\right),\tag{3.5}$$

Then $F(\phi,\sigma)\in L^2(0,T;V^{-3}(\Omega))$ and there exists a constant C depending only on Ω such that

$$\|F(\phi,\sigma)\|_{L^2(0,T;V^{-3}(\Omega))}\leqslant C\|\phi\|^2_{L^4(0,T;H^1_0(\Omega))}+C\|\sigma\|_{C([0,T];L^2(\Omega))}\|\phi\|_{L^2(0,T;H^1_0(\Omega))}.$$

3.2 Compact sets

Here, we state several classical results of compactness that we use in this paper.

Lemma 3.2. *Let $\{f_n\}$ be a bounded sequence in $L^\infty(0,T;H^1_0(\Omega))$ such that $\{\frac{\partial f_n}{\partial t}\}$ is bounded in $L^2(0,T;L^1(\Omega))$. Then $\{f_n\}$ is relatively compact in $C([0,T];L^4(\Omega))$.*

Lemma 3.3. *Let $\{f_n\}$ be a bounded sequence in $L^2(0,T;V^1(\Omega))$ such that $\{\frac{\partial f_n}{\partial t}\}$ is bounded in $L^2(0,T;V^{-k}(\Omega))$ for some $k\in\mathbb{N}^*$. Then $\{f_n\}$ is relatively compact in $L^2([0,T];V^0(\Omega))$.*

The proof of Lemma 3.2 is given in [17, Corollary 8] whereas the proof of Lemma 3.3 is given in [15, pp.57-60].

3.3 Properties of A_σ

If $\sigma\in L^\infty(\Omega)$, then the nonlinear operator A_σ given by (2.3) is defined from $H^1_0(\Omega)$ into $H^{-1}(\Omega)$ as

$$\langle A_\sigma(\phi),\psi\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}:=\int_\Omega\lambda\nabla\phi\cdot\nabla\psi\,\mathrm{dx}+\int_\Omega\sigma\left(\frac{\nabla\phi}{\sqrt{|\nabla\phi|^2+\varepsilon}}\right)\cdot\nabla\psi\,\mathrm{dx}.\tag{3.6}$$

Let us now assume that ϕ is a (weak) solution of $A_\sigma \phi = f$, where $f \in L^2(\Omega)$, i.e.

$$\begin{cases} -\nabla \cdot \left((\lambda + \dfrac{\sigma}{\sqrt{|\nabla \phi|^2 + \epsilon}}) \nabla \phi \right) = f \quad \text{in } \Omega, \\ \phi = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

The following proposition shows that, if σ is regular, then $\phi \in H^2(\Omega)$ and in particular, that the L^2 norm of $A_\sigma(\phi)$ control the H^2 norm of ϕ:

Proposition 3.4 (H^2-regularity of a nonlinear elliptic equation)**.** *Assume that $\sigma \in C^2(\overline{\Omega})$, that $\sigma \geqslant 0$ and that $\partial\Omega$ is C^2. If $\phi \in H_0^1(\Omega)$ is a weak solution of* (3.7)*, then $\phi \in H^2(\Omega)$ and*

$$\|\phi\|_{H^2(\Omega)} \leqslant C \|f\|_{L^2(\Omega)}.$$

The constant C above can be chosen independently of σ.

The proof of the above proposition is quite classical since (3.7) is similar to Euler–Lagrange equations obtained in the calculus of variations. We refer to [8] and for a simple and direct proof of the above proposition, one can adapt the proof of Theorem 1, p.459 in [7].

3.4 Classical results on the transport equation

This subsection is devoted to recalling classical results on the transport equation

$$\begin{cases} \sigma_t + \nabla \cdot (u\ \sigma) = 0, \\ \sigma(0, x) = \sigma_0(x), \end{cases} \tag{3.8}$$

where u is a given divergence-free vector field.

The following result is due to DiPerna and Lions [6] (see also [16, p. 41]).

Proposition 3.5. *Assume that $\{\sigma_n\}$ and $\{u_n\}$ are two sequences such that*

$$\sigma_n \in C([0,T]; L^2(\Omega)), \quad u_n \in L^2(0,T; V(\Omega)),$$

with $\{\sigma_n\}$ bounded in $L^\infty((0,T) \times \Omega)$, $\{u_n\}$ bounded in $L^2(0,T;V(\Omega))$ and with

$$\sigma_{n,t} + \nabla \cdot (\sigma_n u_n) = 0,$$
$$\sigma_n(0) \to \sigma_0 \quad \textit{in} \quad L^2(\Omega),$$
$$u_n \rightharpoonup u \quad \textit{in} \quad L^2(0,T;V(\Omega)) - \textit{weak},$$

for some $\sigma_0 \in L^\infty(\Omega)$, $\sigma_0 \geqslant 0$.

Then for every $1 \leqslant p < \infty$,

$$\sigma_n \to \sigma \quad \textit{strongly in} \quad C([0,T]; L^p(\Omega)),$$

where $\sigma \in L^\infty((0,T) \times \Omega)$ is the unique solution of the problem

$$\begin{cases} \sigma_t + \nabla \cdot (\sigma u) = 0 \quad \textit{in} \quad \mathscr{D}'(\Omega \times (0,T)), \\ \sigma(0) = \sigma_0 \quad \textit{a.e. in} \quad \Omega. \end{cases} \tag{3.9}$$

Let us also recall a result on classical solutions of (3.8) (see, for instance, [16, p.67]).

Proposition 3.6. *Assume*

$$\begin{cases} u \in L^2(0,T; V^6(\Omega)), \\ \sigma_0 \in C^2(\bar{\Omega}), \quad \sigma_0 \geqslant 0. \end{cases}$$

Then (3.8) *admits a unique solution σ. Moreover,*

$$\sigma \in C([0,T]; C^2(\bar{\Omega})), \quad \sigma_t \in L^2(0,T; C^1(\bar{\Omega})), \quad 0 \leqslant \sigma \leqslant \|\sigma_0\|_{C(\bar{\Omega})}.$$

and

$$\|\sigma\|_{C([0,T];C^2(\bar{\Omega}))} + \|\sigma_t\|_{L^2(0,T;C^1(\bar{\Omega}))} \leqslant C(\|\sigma_0\|_{C^2(\bar{\Omega})}, \|u\|_{L^2(0,T;V^6(\Omega))}). \tag{3.10}$$

4 Approximation of the Navier–Stokes system

We consider the following approximation of the Navier–Stokes equations:

$$\begin{cases} u_t + \nu\Delta^6 u + u\cdot\nabla u = \Delta u + \nabla p + f & \text{in } (0,T)\times\Omega, \\ \nabla\cdot u = 0 & \text{in } (0,T)\times\Omega, \\ u = \frac{\partial}{\partial n}u = \Delta u = \frac{\partial}{\partial n}\Delta u = \Delta^2 u = \frac{\partial}{\partial n}\Delta^2 u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0 & \text{in } \Omega \end{cases} \tag{4.1}$$

where $\nu > 0$. This system was used in [15, p. 97] to consider the well-posedness of the Navier–Stokes equations. We consider a similar method in our case. Note that we could take other viscous term $\nu(-\Delta)^m u$ (with suitable boundary conditions) but the power $m = 6$ is sufficient in our case. Here, we show that the above system is well-posed.

Proposition 4.1. *Assume $f \in L^2(0,T;V^{-6}(\Omega))$ and $u_0 \in H(\Omega)$, then (4.1) admits a unique weak solution*

$$u \in L^2(0,T;V^6(\Omega)) \cap H^1(0,T;V^{-6}(\Omega))$$

in the sense that for all $v \in C^1([0,T];V^6(\Omega))$, with $v(T,\cdot) = 0$, the following relation holds true:

$$-\int_{Q^T} u\cdot v_t \,\mathrm{dx}\,\mathrm{dt} - \int_{\Omega} u_0(\cdot)\cdot v(0,\cdot)\,\mathrm{dx} + \nu\int_0^T (u,v)_{V^6(\Omega)}\,\mathrm{dt} - \int_{Q^T} (u\otimes u - \nabla u) : \nabla v\,\mathrm{dx}\,\mathrm{dt}$$
$$= \int_0^T \langle f, v\rangle_{V^{-6}(\Omega),V^6(\Omega)}\,\mathrm{dt}\,. \tag{4.2}$$

Moreover, the following estimate holds true:

$$\|u_t\|_{L^2(0,T;V^{-6}(\Omega))} + \|u\|_{L^2(0,T;V^6(\Omega))} + \|u\|_{C([0,T];H(\Omega))}$$
$$\leqslant C(\nu)\|f\|_{L^2(0,T;V^{-6}(\Omega))} + \|u_0\|_{H(\Omega)}. \tag{4.3}$$

We recall that the spaces $V^6(\Omega)$, $H(\Omega)$ and $V^{-6}(\Omega)$ are defined in Subsection 2.1. In (4.2), we use the following inner product of $V^6(\Omega)$:

$$(u,v)_{V^6(\Omega)} = \int_{\Omega} \Delta^3 u\cdot\Delta^3 u\,\mathrm{dx} \quad \forall u,v \in V^6(\Omega).$$

The norm associated to this inner product is equivalent to the canonical norm of $H^6(\Omega)$. Since the proof of Proposition 4.1 is very similar to the proof of existence of the classical Navier–Stokes system, we only give here the main steps of the proof (see [15, pp. 75-77] for more details). We first state a classical lemma which can be found for instance in [15, p. 74, Corollary 6.1].

Lemma 4.2. *The operator $A = \nu\Delta^6 \in \mathscr{L}(V^6(\Omega),V^{-6}(\Omega))$ is self-adjoint positive with compact resolvent. Moreover, $H(\Omega)$ admits an orthonormal basis $\{\omega_k\}_{k=1}^{\infty} \subset V^6(\Omega)$ of eigenvectors of A.*

Proof of Proposition 4.1. We use a classical Galerkin method: for any $k \in \mathbb{N}^*$, we denote by $P_k : H(\Omega) \to \operatorname{span}\{\omega_1,\cdots,\omega_k\}$ the orthogonal projector. Using Lemma 4.2, we deduce that $P_k \in \mathscr{L}(V^6(\Omega),V^6(\Omega))$ and

$$\|P_k\|_{\mathscr{L}(V^6(\Omega),V^6(\Omega))} \leqslant 1. \tag{4.4}$$

By a duality argument, we have $P_k^* \in \mathscr{L}(V^{-6}(\Omega),V^{-6}(\Omega))$ and

$$\|P_k^*\|_{\mathscr{L}(V^{-6}(\Omega),V^{-6}(\Omega))} \leqslant 1. \tag{4.5}$$

For any $k \in \mathbb{N}^*$, there exists a unique solution

$$u_k(t) = \sum_{j=1}^{k} g_{jk}(t)\omega_j(x),$$

of the problem

$$u_k' + P_k^* A u_k = -P_k^*(u_k \cdot \nabla u_k) + P_k^* f. \tag{4.6}$$

$$u_k(0) = P_k u_0. \tag{4.7}$$

As for the classical Navier-Stokes equations, we have the following a priori estimate

$$\frac{1}{2}\frac{d}{dt}\|u_k\|_{H(\Omega)}^2 + \nu\|u_k\|_{V^6(\Omega)}^2 + \|u_k\|_{V^1(\Omega)}^2 \leqslant \langle f, u_k\rangle_{V^{-6}(\Omega),V^6(\Omega)}, \tag{4.8}$$

and thus

$$\|u_k\|_{L^2(0,T;V^6(\Omega))} + \|u_k\|_{L^\infty(0,T;H(\Omega))} \leqslant C(\nu)\|f\|_{L^2(0,T;V^{-6}(\Omega))} + \|u_0\|_{H(\Omega)}. \tag{4.9}$$

Using Sobolev embedding and Hölder's inequality, one can show that

$$\|P_k^*(u_k \cdot \nabla u_k)\|_{L^2(0,T;V^{-6}(\Omega))} \leqslant C\|u_k\|_{L^\infty(0,T;H(\Omega))}\|u_k\|_{L^2(0,T;V^6(\Omega))}, \tag{4.10}$$

which, combined with (4.9), implies

$$\|u_k'\|_{L^2(0,T;V^{-6}(\Omega))} \leqslant C(\nu)\|f\|_{L^2(0,T;V^{-6}(\Omega))} + \|u_0\|_{H(\Omega)}. \tag{4.11}$$

We deduce from (4.9) and (4.11) that, up to the extraction of a subsequence,

$$u_k \rightharpoonup u \quad L^2(0,T;V^6(\Omega)) - \text{weak} \tag{4.12}$$

and

$$u_k' \rightharpoonup u' \quad L^2(0,T;V^{-6}(\Omega)) - \text{weak}. \tag{4.13}$$

Combining the above two statements with Lemma 3.3, we can pass to the limit. By weak sequential lower semicontinuity (see, for instance, [18, p. 235]), we obtain (4.3) by combining (4.12), (4.13), (4.9) and (4.11). The uniqueness can be done in a classical way by using the regularity of u. $\square$

Let us end this section by a result of continuity: assume

$$f_k \to f \quad \text{in } L^2(0,T;V^{-6}(\Omega)) \tag{4.14}$$

and assume $u_0 \in H(\Omega)$. We consider for all k the unique weak solution

$$u_k \in L^2(0,T;V^6(\Omega)) \cap H^1(0,T;V^{-6}(\Omega))$$

of

$$\begin{cases} (u_k)_t + \nu\Delta^6 u_k + u_k \cdot \nabla u_k = \Delta u_k + \nabla p_k + f_k & \text{in } (0,T)\times\Omega, \\ \nabla \cdot u_k = 0 & \text{in } (0,T)\times\Omega, \\ u_k = \frac{\partial}{\partial n}u_k = \Delta u_k = \frac{\partial}{\partial n}\Delta u_k = \Delta^2 u_k = \frac{\partial}{\partial n}\Delta^2 u_k = 0 & \text{on } (0,T)\times\partial\Omega, \\ u_k(0,\cdot) = u_0 & \text{in } \Omega. \end{cases} \tag{4.15}$$

We also denote by u the unique weak solution of (4.1)

Proposition 4.3. *Assume* (4.14) *and let us consider u_k and u defined as above. Then*

$$u_k \to u \quad \textit{in } L^2(0,T;V^6(\Omega)) \cap L^\infty(0,T;H(\Omega)).$$

The proof of the above result is classical and we skip it.

5 Resolution of the phase-field equation

Now we turn to the phase-field equation (1.2c). More precisely, we consider the initial-boundary value problem of the following nonlinear parabolic equation:

$$\begin{cases} \phi_t + A_\sigma(\phi) = -u \cdot \nabla\phi - W'(\phi) & \text{in } (0,T) \times \Omega, \\ \phi = 0 & \text{on } (0,T) \times \partial\Omega, \\ \phi(0,\cdot) = \phi_0 & \text{in } \Omega, \end{cases} \tag{5.1}$$

where $A_\sigma(\phi)$ is defined by (3.6). The following proposition shows that $A_\sigma(\phi)$ is a monotone operator provided that σ is non-negative.

Proposition 5.1. *There exists a positive constant C depending only on Ω such that*

$$\|A_\sigma(\phi)\|_{H^{-1}(\Omega)} \leqslant \lambda\|\phi\|_{H_0^1(\Omega)} + C\|\sigma\|_{L^\infty(\Omega)} \quad (\phi \in H_0^1(\Omega), \sigma \in L^\infty(\Omega)). \tag{5.2}$$

Moreover, if $\sigma \in L^\infty(\Omega)$ and $\sigma \geqslant 0$ a.e. in Ω, then

$$\langle A_\sigma(\phi_1) - A_\sigma(\phi_2), \phi_1 - \phi_2\rangle \geqslant \lambda\|\phi_1 - \phi_2\|^2_{H_0^1(\Omega)} \quad (\phi_1, \phi_2 \in H_0^1(\Omega)). \tag{5.3}$$

Proof. From (3.6) and from (2.4), we deduce

$$\langle A_\sigma(\phi), \psi\rangle := \int_\Omega \lambda\nabla\phi \cdot \nabla\psi \,\mathrm{dx} + \int_\Omega J_\sigma(\phi) \cdot \nabla\psi \,\mathrm{dx}\,. \tag{5.4}$$

with

$$\|J_\sigma(\phi)\|_{L^2(\Omega)} \leqslant C\|\sigma\|_{L^\infty(\Omega)}.$$

The above inequality and (5.4) yield (5.2).

Assume $\sigma \in L^\infty(\Omega)$ and $\sigma \geqslant 0$ a.e. in Ω, and let us consider ϕ_1, $\phi_2 \in H_0^1(\Omega)$. Using (5.4), we obtain

$$\begin{aligned} &\langle A_\sigma(\phi_1) - A_\sigma(\phi_2), \phi_1 - \phi_2\rangle_{H_0^1(\Omega), H^{-1}(\Omega)} \\ &\qquad = \lambda\|\nabla\phi_1 - \nabla\phi_2\|^2_{L^2(\Omega)} + \int_\Omega (J_\sigma(\phi_1) - J_\sigma(\phi_2)) \cdot (\nabla\phi_1 - \nabla\phi_2)\,\mathrm{dx}\,. \end{aligned} \tag{5.5}$$

Since for all $\alpha > 0$, the function $z \mapsto z/\sqrt{z^2+\alpha}$ is increasing on $\mathbb{R}$, it follows

$$\left(\frac{x}{\sqrt{|x|^2+\epsilon}} - \frac{\bar{x}}{\sqrt{|\bar{x}|^2+\epsilon}}\right) \cdot (x - \bar{x}) \geqslant 0 \quad (x, \bar{x} \in \mathbb{R}^3).$$

The above relation combined with the definition (2.4) of J_σ and the hypothesis $\sigma \geqslant 0$ yields

$$(J_\sigma(\phi_1) - J_\sigma(\phi_2)) \cdot (\nabla\phi_1 - \nabla\phi_2) \geqslant 0 \quad \text{in } \Omega.$$

The above inequality and (5.5) imply (5.3). □

The following result is essentially contained in [15, pp. 159-161] which can be considered as the application of monotone operator theory to parabolic equation.

Proposition 5.2 (Monotone method)**.** *Let $\sigma_n, \sigma \in L^\infty((0,T)\times\Omega)$, $\phi_n, \phi \in L^\infty(0,T; H_0^1(\Omega))$ and $\chi \in L^2(0,T; H^{-1}(\Omega))$. Assume*

$$\begin{cases} \sigma_n \to \sigma \quad C([0,T]; L^p(\Omega)) - \textit{strong}, \ \forall p \in (1,\infty), \\ \sigma_n, \sigma \geqslant 0 \quad \textit{a.e.} \quad (t,x) \in [0,T] \times \Omega, \\ \phi_n \to \phi \quad C([0,T]; L^4(\Omega)) - \textit{strong}, \\ \phi_n \rightharpoonup \phi \quad L^\infty(0,T; H_0^1(\Omega)) - \textit{weak*}, \\ A_{\sigma_n}(\phi_n) \rightharpoonup \chi \quad L^2(0,T; H^{-1}(\Omega)) - \textit{weak}, \end{cases} \tag{5.6}$$

$$\frac{1}{2}\int_{\Omega}|\phi_n(T)|^2\,\mathrm{dx}+\int_0^T\langle A_{\sigma_n}(\phi_n),\phi_n\rangle\,\mathrm{dt}+\int_{Q^T}\phi_n W'(\phi_n)\,\mathrm{dx}\,\mathrm{dt}=\frac{1}{2}\int_{\Omega}|\phi_n(0)|^2\,\mathrm{dx}, \tag{5.7}$$

$$\frac{1}{2}\int_{\Omega}|\phi(T)|^2\,\mathrm{dx}+\int_0^T\langle \chi,\phi\rangle\,\mathrm{dt}+\int_{Q^T}\phi W'(\phi)\,\mathrm{dx}\,\mathrm{dt}=\frac{1}{2}\int_{\Omega}|\phi(0)|^2\,\mathrm{dx}\,. \tag{5.8}$$

Then $A_\sigma(\phi)=\chi$ *and*

$$\nabla\phi_n\to\nabla\phi \quad \textit{strongly in} \quad L^2(0,T;L^2(\Omega)). \tag{5.9}$$

Proof. Let $\hat\phi\in L^2(0,T;H_0^1(\Omega))$. We first claim that, under condition (5.6), we have

$$\int_0^T\langle A_{\sigma_n}(\hat\phi),\phi_n-\hat\phi\rangle\,\mathrm{dt}\to\int_0^T\langle A_{\sigma}(\hat\phi),\phi-\hat\phi\rangle\,\mathrm{dt}\,. \tag{5.10}$$

Indeed, from (5.4), we deduce

$$\int_0^T\langle A_{\sigma_n}(\hat\phi),\phi_n-\hat\phi\rangle\,\mathrm{dt}\;=\int_{Q^T}(\lambda\nabla\hat\phi+J_{\sigma_n}(\hat\phi))\cdot(\nabla\phi_n-\nabla\hat\phi)\,\mathrm{dx}\,\mathrm{dt}\,. \tag{5.11}$$

Combining the first statement of (5.6) with Lebesgue's Dominated convergence theorem, we have

$$J_{\sigma_n}(\hat\phi)\to J_\sigma(\hat\phi) \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)),$$

which combined with (5.11) and the fourth statement of (5.6) indicates that

$$\lim_{n\to\infty}\int_0^T\langle A_{\sigma_n}(\hat\phi),\phi_n-\hat\phi\rangle\,\mathrm{dt}=\int_{Q^T}(\lambda\nabla\hat\phi+J_{\sigma}(\hat\phi))\cdot(\nabla\phi-\nabla\hat\phi)\,\mathrm{dx}\,\mathrm{dt},$$

which implies (5.10).

Using (1.4), we deduce that $|\phi_n W'(\phi_n)|\leqslant C(|\phi_n|^4+1)$, which, combined with the third statement of (5.6), implies

$$\int_{Q^T}\phi_n W'(\phi_n)\,\mathrm{dx}\,\mathrm{dt}\to\int_{Q^T}\phi W'(\phi)\,\mathrm{dx}\,\mathrm{dt}\,. \tag{5.12}$$

Let us set

$$X_n=\int_0^T\langle A_{\sigma_n}(\phi_n)-A_{\sigma_n}(\hat\phi),\phi_n-\hat\phi\rangle\,\mathrm{dt}\,. \tag{5.13}$$

Substituting (5.7) into (5.13) yields

$$X_n=\frac{1}{2}\int_{\Omega}|\phi_n(0)|^2\,\mathrm{dx}-\frac{1}{2}\int_{\Omega}|\phi_n(T)|^2\,\mathrm{dx}-\int_{Q^T}\phi_n W'(\phi_n)\,\mathrm{dx}\,\mathrm{dt}$$
$$-\int_0^T\langle A_{\sigma_n}(\phi_n),\hat\phi\rangle\,\mathrm{dt}-\int_0^T\langle A_{\sigma_n}(\hat\phi),\phi_n-\hat\phi\rangle\,\mathrm{dt}\,.$$

By combining the above equation with (5.6), (5.10) and (5.12) implies that

$$\lim_{n\to\infty} X_n = \frac{1}{2}\int_\Omega |\phi(0)|^2\,\mathrm{dx} - \frac{1}{2}\int_\Omega |\phi(T)|^2\,\mathrm{dx} - \int_{Q^T} \phi W'(\phi)\,\mathrm{dx}\,\mathrm{dt} - \int_0^T \langle \chi, \hat{\phi}\rangle\,\mathrm{dt} - \int_0^T \langle A_\sigma(\hat{\phi}), \phi - \hat{\phi}\rangle\,\mathrm{dt}\,.$$

Substituting (5.8) into the above equality implies

$$\lim_{n\to\infty} X_n = \int_0^T \langle \chi - A_\sigma(\hat{\phi}), \phi - \hat{\phi}\rangle\,\mathrm{dt}\,. \tag{5.14}$$

Now, from the definition (5.13) of X_n and from (5.3), we deduce from the above equality

$$\int_0^T \langle \chi - A_\sigma(\hat{\phi}), \phi - \hat{\phi}\rangle\,\mathrm{dt} \geqslant \limsup_{n\to\infty} \lambda \|\phi_n - \hat{\phi}\|^2_{L^2(0,T;H_0^1(\Omega))}. \tag{5.15}$$

Taking $\hat{\phi} = \phi$ in (5.15) gives

$$\nabla\phi_n \to \nabla\phi \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)).$$

Taking $\hat{\phi} = \phi - \alpha w$, with $\alpha > 0$ and $w \in L^2(0,T;H_0^1(\Omega))$ in (5.15) implies

$$\alpha \int_0^T \langle \chi - A_\sigma(\phi - \alpha w), w\rangle\,\mathrm{dt} \geqslant 0$$

and thus

$$\int_0^T \langle \chi - A_\sigma(\phi - \alpha w), w\rangle\,\mathrm{dt} \geqslant 0.$$

Let us make α tend to 0:

$$\int_0^T \langle \chi - A_\sigma(\phi), w\rangle\,\mathrm{dt} \geqslant 0, \quad \forall w \in L^2(0,T;H_0^1(\Omega)),$$

which implies, by a density argument, $\chi = A_\sigma(\phi)$. □

With the help of the above proposition, we can state and prove the following proposition which shows that (5.1) admits a unique strong solution provided that σ, u and the initial data ϕ_0 are regular enough.

Proposition 5.3. *Assume*

$$\begin{cases} u \in L^2(0,T;H^2(\Omega)\cap V(\Omega)), \\ \sigma \in C([0,T];C^2(\bar{\Omega})), \\ \sigma(t,x) \geqslant 0 \quad a.e. \quad (t,x) \in [0,T]\times\Omega), \\ \sigma_t \in L^2(0,T;L^2(\Omega)), \\ \phi_0 \in H_0^1(\Omega). \end{cases} \tag{5.16}$$

and assume that there exists a positive constant C_0 such that

$$\|u\|_{L^2(0,T;H^2(\Omega)\cap V(\Omega))} + \|\sigma\|_{C([0,T];C^2(\bar{\Omega}))} + \|\sigma_t\|_{L^2(0,T;L^2(\Omega))} + \|\phi_0\|_{H_0^1(\Omega)} \leqslant C_0,$$

then equation (5.1) *admits a unique solution*

$$\phi \in L^2(0,T;H^2(\Omega)) \cap C([0,T];H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega)).$$

Furthermore, there exists a positive constant $C = C(C_0,\Omega,T)$ such that

$$\|\phi\|_{L^2(0,T;H^2(\Omega)} + \|\phi\|_{C([0,T];H_0^1(\Omega))} + \|\phi\|_{H^1(0,T;L^2(\Omega))} \leqslant C. \tag{5.17}$$

Proof. The proof is divided into several steps.
1st step: Construction of approximate solutions
Let $\{w_i\}_{i=1}^\infty$ be an orthonormal base of $H_0^1(\Omega)$. For all $m \geqslant 1$, we consider the L^2 projection ϕ_{0m} of ϕ_0 on $\operatorname{span}\{w_1,\ldots,w_m\}$ and ϕ_m the solution of

$$\begin{cases} \phi_m(t) \in \operatorname{span}\{w_1,\ldots,w_m\} \quad (t \geqslant 0), \\ (\phi_m'(t), w_j) + \langle A_\sigma(\phi_m(t)), w_j\rangle = -(W'(\phi_m(t)) + u\cdot\nabla\phi_m(t), w_j) \quad (1 \leqslant j \leqslant m), \\ \phi_m(0) = \phi_{0m}. \end{cases} \tag{5.18}$$

We can derive in a classical way the following estimate:

$$\begin{aligned} &\int_\Omega |\phi_m'|^2 \,\mathrm{dx} + \partial_t \int_\Omega \left(\frac{\lambda}{2}|\nabla\phi_m|^2 + W(\phi_m) + \sigma\sqrt{|\nabla\phi_m|^2+\epsilon}\right)\mathrm{dx} \\ &= -\int_\Omega (u\cdot\nabla\phi_m)\phi_m' \,\mathrm{dx} + \int_\Omega \sigma_t\sqrt{|\nabla\phi_m|^2+\epsilon}\ \mathrm{dx} \\ &\leqslant \frac{1}{2}\|\phi_m'\|^2_{L^2(\Omega)} + \frac{1}{2}\|u\cdot\nabla\phi_m\|^2_{L^2(\Omega)} + \frac{1}{2}\|\sigma_t\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla\phi_m\|^2_{L^2(\Omega)} + \frac{\epsilon}{2}|\Omega|. \end{aligned} \tag{5.19}$$

Using a Sobolev embedding, we have

$$\|u\cdot\nabla\phi_m\|^2_{L^2(\Omega)} \leqslant C\|u\|^2_{H^2(\Omega)}\|\nabla\phi_m\|^2_{L^2(\Omega)}.$$

Combining the above estimate with (5.19) and Gronwall inequality implies that

$$\begin{aligned} &\int_{Q^T} |\phi_m'|^2 \,\mathrm{dx}\,\mathrm{dt} + \max_{t\in[0,T]}\int_\Omega \left(\frac{\lambda}{2}|\nabla\phi_m(t)|^2 + W(\phi_m(t)) + \sigma\sqrt{|\nabla\phi_m(t)|^2+\epsilon}\right)\mathrm{dx} \\ &\qquad\leqslant C\exp\left(C\int_0^T \|u(s)\|^2_{H^2(\Omega)}\mathrm{d}s\right) \\ &\qquad\times\left(\int_\Omega \left(\frac{\lambda}{2}|\nabla\phi_0|^2 + W(\phi_0) + \sigma_0\sqrt{|\nabla\phi_0|^2+\epsilon}\right)\mathrm{dx} + \epsilon + \|\sigma_t\|^2_{L^2(0,T;L^2(\Omega))}\right). \end{aligned} \tag{5.20}$$

This yields in particular that

$$\{\phi_m\} \quad \text{is bounded in} \quad H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)). \tag{5.21}$$

Using (5.2) and the second condition of (5.16), we deduce from the above relation that

$$A_\sigma(\phi_m) \quad \text{is bounded in} \quad L^2(0,T;H^{-1}(\Omega)). \tag{5.22}$$

Combining (5.21) and (5.22), we conclude that, up to a subsequence,

$$\begin{cases} \phi_m \rightharpoonup \phi & \text{in } L^\infty(0,T;H_0^1(\Omega)) - \text{weak*}, \\ \phi_m' \rightharpoonup \phi' & \text{in } L^2(0,T;L^2(\Omega)) - \text{weak}, \\ A_\sigma(\phi_n) \rightharpoonup \chi & \text{in } L^2(0,T;H^{-1}(\Omega)) - \text{weak}. \end{cases} \tag{5.23}$$

Moreover, combining (5.21) with Lemma 3.2, up to the extraction of a subsequence, we have

$$\phi_m \to \phi \quad \text{strongly in} \quad C([0,T];L^4(\Omega)). \tag{5.24}$$

In particular,

$$W'(\phi_m) \to W'(\phi) \quad \text{in } C([0,T];L^{4/3}(\Omega)).$$

2nd step: Passing to the limit
Using Step 1 of the proof and Sobolev embeddings, we deduce that

$$\phi_t + \chi + W'(\phi) + u \cdot \nabla\phi = 0 \quad \text{in } L^2(0,T;H^{-1}(\Omega)).$$

In particular, multiplying the above equation by $\phi \in L^\infty(0,T;H_0^1(\Omega))$, we obtain that ϕ satisfies (5.8). Applying Proposition 5.2, we conclude that $\chi = A_\sigma(\phi)$. Therefore, we deduce the existence of a weak solution of (5.1):

$$\phi \in H^1(0,T;L^2(\Omega)) \cap L^\infty(0,T;H_0^1(\Omega)). \tag{5.25}$$

From Sobolev embeddings, we remark that

$$f := \phi_t + W'(\phi) + u \cdot \nabla\phi \in L^2(0,T;L^2(\Omega)$$

and from (5.1), we have, for almost every $t \in (0,T)$,

$$\begin{cases} \nabla \cdot \left((\lambda + \dfrac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi \right) = f \quad \text{in } \Omega, \\ \phi = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Applying Proposition 3.4 yields that

$$\|\phi\|_{L^2(0,T;H^2(\Omega))} \leqslant C\|f\|_{L^2(0,T;L^2(\Omega))}, \tag{5.26}$$

where C is a positive constant depending only on $\|\sigma\|_{C([0,T],C^2(\bar{\Omega}))}$. This concludes the proof. □

6 Viscous Approximation of (1.2)–(1.3)

In this section, we consider a system approximating (1.2)–(1.3), obtained by adding a viscous term (and suitable boundary conditions). More precisely, we study the following initial-boundary value problem:

$$\begin{cases} u_t + \nu\Delta^6 u + u \cdot \nabla u = \Delta u + \nabla p - \nabla \cdot \left((\lambda + \dfrac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi \otimes \nabla\phi \right) & \text{(6.1a)} \\ \nabla \cdot u = 0 & \text{(6.1b)} \\ \phi_t - \nabla \cdot \left((\lambda + \dfrac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi \right) = -W'(\phi) - u \cdot \nabla\phi & \text{(6.1c)} \\ \sigma_t + u \cdot \nabla\sigma = 0 & \text{(6.1d)} \end{cases}$$

in $(0,T)\times\Omega$ and

$$\begin{cases} u = \dfrac{\partial}{\partial n}u = \Delta u = \dfrac{\partial}{\partial n}\Delta u = \Delta^2 u = \dfrac{\partial}{\partial n}\Delta^2 u = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ \phi = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0 \quad \text{in } \Omega, \quad \phi(0,\cdot) = \phi_0 \quad \text{in } \Omega, \quad \sigma(0,\cdot) = \sigma_0 \quad \text{in } \Omega. \end{cases} \tag{6.2}$$

Remark 6.1. *Such a viscosity method is quite classical (see, for instance, [15]) and was used recently (see, for instance, [3] and [9]).*

We show an a priori estimate for the above system which is similar to (2.12). Let us recall that $E(t)$ is defined by (2.13). If

$$\begin{cases} u \in L^2(0,T;V^6(\Omega)) \cap H^1(0,T;V^{-6}(\Omega)), \\ \phi \in L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega)) \cap H^1(0,T;L^2(\Omega)), \\ \sigma \in H^1(0,T;C^1(\bar{\Omega})) \cap C([0,T];C^2(\bar{\Omega})), \end{cases} \tag{6.3}$$

we say that (u,ϕ,σ) is a solution of (6.1)–(6.2) if (6.1b)–(6.1d), (6.2) are satisfied in a strong sense and if

$$-\int_{Q^T} u\cdot v_t \,\mathrm{dx}\,\mathrm{dt} - \int_\Omega u_0(\cdot)v(0,\cdot)\,\mathrm{dx} + \nu\int_{Q^T}\Delta^3 u\cdot\Delta^3 v\,\mathrm{dx}\,\mathrm{dt} - \int_{Q^T}(u\otimes u - \nabla u):\nabla v\,\mathrm{dx}\,\mathrm{dt}$$
$$= \int_0^T \langle F(\phi,\sigma), v\rangle_{V^{-3}(\Omega),V^3(\Omega)}\,\mathrm{dt}, \quad \text{for all} \quad v\in C^1([0,T];V^6(\Omega)),\ v(T,\cdot)=0. \tag{6.4}$$

We recall that $F(\phi,\sigma)$ is defined in Proposition 3.1.

Proposition 6.2 (A priori estimates)**.** *Assume* (u,ϕ,σ) *satisfies* (6.3) *and is a solution of system* (6.1)–(6.2). *Then we have the following a priori estimate*

$$E(t) + \int_{Q^t}\left(|A_\sigma(\phi)+W'(\phi)|^2 + |\nabla u|^2 + \nu|\Delta^3 u|^2\right)\mathrm{dx}\,\mathrm{dt} = E(0), \quad \textit{a.e. } t\in(0,T). \tag{6.5}$$

Proof. The proof follows the lines of Subsection 2.2. The only difference comes from the fact that u satisfies only a weak formulation (6.4). To obtain an estimate similar to (2.10), we can not take u as a test function in (6.4). To overcome this difficulty, we take a sequence $u_n\in C^1([0,T];V^6(\Omega))$, with $u_n(T,\cdot)=0$ and

$$u_n \to u 1_{[0,t]} \quad \text{in } L^2(0,T;V^6(\Omega)).$$

We also use the decomposition (2.7) of $F(\phi,\sigma)$, so that

$$-\int_{Q^T} u\cdot u_{nt}\,\mathrm{dx}\,\mathrm{dt} - \int_\Omega u_0(\cdot)u_n(0,\cdot)\,\mathrm{dx} + \nu\int_{Q^T}\Delta^3 u\cdot\Delta^3 u_n\,\mathrm{dx}\,\mathrm{dt} - \int_{Q^T}(u\otimes u-\nabla u):\nabla u_n\,\mathrm{dx}\,\mathrm{dt}$$
$$= \int_0^T \langle A_\sigma(\phi)\nabla\phi + \left(\sqrt{|\nabla\phi|^2+\epsilon}\right)\nabla\sigma, u_n\rangle_{V^{-3}(\Omega),V^3(\Omega)}\,\mathrm{dt}\,. \tag{6.6}$$

From the regularity (6.3), we deduce $A_\sigma(\phi)\nabla\phi\in L^2(0,T;L^1(\Omega))$ and $\nabla\sigma\sqrt{|\nabla\phi|^2+\epsilon}\in L^2(Q^T)$.

Moreover, an integration by parts shows that

$$-\int\limits_{Q^T} u\cdot u_{nt}\,\mathrm{dx}\,\mathrm{dt}-\int\limits_{\Omega} u_0(\cdot)u_n(0,\cdot)\,\mathrm{dx}=\int\limits_0^T\langle u_t,u_n\rangle_{V^{-6}(\Omega),V^6(\Omega)}\,\mathrm{d}\,\tau$$

$$\to\frac{1}{2}\int\limits_{\Omega}|u(t,\cdot)|^2\,\mathrm{dx}-\frac{1}{2}\int\limits_{\Omega}|u_0|^2\,\mathrm{dx}\,.$$

As a consequence, we deduce

$$\frac{1}{2}\int\limits_{\Omega}|u(t,\cdot)|^2\,\mathrm{dx}+\|\nabla u\|^2_{L^2(0,t;L^2(\Omega))}+\nu\|\Delta^3u\|^2_{L^2(0,t;L^2(\Omega))}$$

$$-\int\limits_{Q^t}A_\sigma(\phi)\nabla\phi\cdot u\,\mathrm{dx}\,\mathrm{d}\,\tau-\int\limits_{Q^t}\nabla\sigma\cdot u\sqrt{|\nabla\phi|^2+\epsilon}\,\mathrm{dx}\,\mathrm{d}\,\tau$$

$$=\frac{1}{2}\int\limits_{\Omega}|u_0|^2dx.\quad a.e.\ t\in(0,T).\qquad(6.7)$$

Using the above relation, we can follow the calculation of Subsection 2.2 to deduce the result. □

The main result of this section is the existence and uniqueness of a solution of (6.1)–(6.2).

Proposition 6.3. *Assume*

$$u_0\in H(\Omega),\quad \phi_0\in H^1_0(\Omega),\quad \sigma_0\in C^2(\bar\Omega),\ \sigma_0\geqslant 0.\qquad(6.8)$$

Then, for every $T>0$, system (6.1)–(6.2) *admits a unique solution*

$$\begin{cases} u\in L^2(0,T;V^6(\Omega))\cap C([0,T];H(\Omega)),\\ \phi\in L^2(0,T;H^2(\Omega)\cap H^1_0(\Omega))\cap C([0,T];H^1_0(\Omega))\cap H^1(0,T;L^2(\Omega)),\\ \sigma\in C([0,T];C^2(\bar\Omega))\cap H^1(0,T;C^1(\bar\Omega)).\end{cases}\qquad(6.9)$$

Proof. The proof is divided into several steps.
1st step: Local existence

Let us consider $\hat T\in(0,T)$ and let us set

$$\mathscr{E}=\{(\phi,\sigma)\in L^4(0,\hat T;H^1_0(\Omega))\times C([0,\hat T];L^2(\Omega))\},$$

$$\mathscr{B}_R=\{(\phi,\sigma)\in\mathscr{E}\ ;\ \|\phi\|_{L^4(0,\hat T;H^1_0(\Omega))}+\|\sigma\|_{C([0,\hat T];L^2(\Omega))}\leqslant R\}.\qquad(6.10)$$

We define the map by

$$\mathscr{F}(\bar\phi,\bar\sigma)=(\phi,\sigma),\quad\forall(\bar\phi,\bar\sigma)\in\mathscr{B}_R$$

where (u,ϕ,σ) is the solutions of the following nonlinear system

$$\begin{cases} u_t+\nu\Delta^6u+u\cdot\nabla u=\Delta u+\nabla p+F(\bar\phi,\bar\sigma) & (6.11a)\\ \nabla\cdot u=0 & (6.11b)\\ \phi_t-\nabla\cdot\left((\lambda+\dfrac{\sigma}{\sqrt{|\nabla\phi|^2+\epsilon}})\nabla\phi\right)=-W'(\phi)-u\cdot\nabla\phi & (6.11c)\\ \sigma_t+u\cdot\nabla\sigma=0 & (6.11d)\end{cases}$$

in $(0,T)\times\Omega$ and

$$\begin{cases} u = \dfrac{\partial}{\partial n}u = \Delta u = \dfrac{\partial}{\partial n}\Delta u = \Delta^2 u = \dfrac{\partial}{\partial n}\Delta^2 u = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ \phi = 0 \quad \text{on } (0,T)\times\partial\Omega, \\ u(0,\cdot) = u_0 \quad \text{in } \Omega, \quad \phi(0,\cdot)=\phi_0 \quad \text{in } \Omega, \quad \sigma(0,\cdot)=\sigma_0 \quad \text{in } \Omega. \end{cases} \tag{6.12}$$

In (6.11a), $F(\bar{\phi},\bar{\sigma})$ is the nonlinear operator defined by (3.5).

In order to prove that the system (6.11)–(6.12) admits a unique solution for $(\bar{\phi},\bar{\sigma})\in\mathscr{B}_R$, we consider several steps:

- We apply Proposition 3.1 which implies that $F(\bar{\phi},\bar{\sigma})\in L^2(0,\hat{T};V^{-3}(\Omega))$ with
$$\|F(\bar{\phi},\bar{\sigma})\|_{L^2(0,\hat{T};V^{-3}(\Omega))} \leqslant CR^2. \tag{6.13}$$
- We use Proposition 4.1 to obtain the existence and uniqueness of u solution of (6.11a)–(6.11b), (6.12). Morever from (6.13), we deduce
$$\|u\|_{L^2(0,\hat{T};V^6(\Omega))} + \|u\|_{C([0,\hat{T}];H(\Omega))} \leqslant C(R,\|u_0\|_{H(\Omega)}). \tag{6.14}$$
- We solve the transport equation (6.11d), (6.12) by applying Proposition 3.6. The solution σ satisfies
$$\|\sigma\|_{C([0,\hat{T}];C^2(\bar{\Omega}))} + \|\sigma_t\|_{L^2(0,\hat{T};C^1(\bar{\Omega}))} \leqslant C(R,\|\sigma_0\|_{C^2(\bar{\Omega})},\|u_0\|_{H(\Omega)},T). \tag{6.15}$$
- Finally, Proposition 5.3 implies the existence and uniqueness of a solution ϕ to (6.11c), (6.12) with
$$\|\phi\|_{L^2(0,\hat{T};H^2(\Omega))} + \|\phi\|_{C([0,\hat{T}];H^1_0(\Omega))} + \|\phi\|_{H^1(0,\hat{T};L^2(\Omega))} \leqslant C(R,\|\sigma_0\|_{C^2(\bar{\Omega})},\|\phi_0\|_{H^1_0(\Omega)},\|u_0\|_{H(\Omega)},T). \tag{6.16}$$

Using the above construction, we can show that $\mathscr{F}:\mathscr{B}_R\to\mathscr{E}$ is well-defined and $\mathscr{F}(\mathscr{B}_R)$ is relatively compact. Furthermore, using (6.15) yields

$$\|\phi\|_{L^4(0,\hat{T};H^1_0(\Omega))} \leqslant \hat{T}^{\frac14}\|\phi\|_{C([0,\hat{T}];H^1_0(\Omega))} \leqslant \hat{T}^{\frac14}C(R,\|\phi_0\|_{H^1_0(\Omega)},\|u_0\|_{H(\Omega)},\|\sigma_0\|_{C^2(\bar{\Omega})},T),$$

and we can show that

$$\|\sigma\|_{C([0,\hat{T}];L^2(\Omega))} = \|\sigma_0\|_{L^2(\Omega)}.$$

It follows from the above two inequalities that, by taking

$$R \geqslant 2\|\sigma_0\|_{L^2(\Omega)},$$

and by taking $\hat{T}$ small enough, $\mathscr{F}$ maps $\mathscr{B}_R$ into itself. Combining this fact with continuity of $\mathscr{F}$ (Proposition 6.4 below) permits to apply Schauder's fixed point theorem and shows that $\mathscr{F}$ admits a fixed point.

2nd step: Global existence

From the first step, one has to show that

$$t\mapsto \|u(t)\|_{H(\Omega)} + \|\sigma(t)\|_{C^2(\bar{\Omega})} + \|\phi(t)\|_{H^1_0(\Omega)} \tag{6.17}$$

does not blow up in finite time. Since the solution (u, ϕ, σ) satisfy (6.9), we deduce from Proposition 6.2 that (u, ϕ, σ) verifies the estimate (6.5):

$$\int_{\Omega} \left(\frac{1}{2}|u(t)|^2 + \frac{\lambda}{2}|\nabla\phi(t)|^2 + W(\phi(t)) + \sigma\sqrt{|\nabla\phi(t)|^2+\epsilon}\right) \mathrm{dx}$$

$$+ \int_{Q^t} \left(|A_\sigma(\phi) + W'(\phi)|^2 + |\nabla u|^2 + \nu|\Delta^3 u|^2\right) \mathrm{dx\,dt}$$

$$= \int_{\Omega} \left(\frac{1}{2}|u_0|^2 + \frac{\lambda}{2}|\nabla\phi_0|^2 + W(\phi_0) + \sigma_0\sqrt{|\nabla\phi_0|^2+\epsilon}\right) \mathrm{dx} \quad a.e.\ t \in (0,T).$$

In particular, it only remains to show that $\|\sigma(t)\|_{C^2(\bar{\Omega})}$ does not blow up in finite time. This is a consequence of Proposition 3.6 and from the above estimate on u. □

Proposition 6.4. *Assume $\hat{T} > 0$. The mapping $\mathscr{F} : \mathscr{B}_R \to \mathscr{E}$ defined by* (6.11), (6.12) *is continuous.*

Proof. Assume

$$(\bar{\phi}_n, \bar{\sigma}_n) \to (\bar{\phi}, \bar{\sigma}) \quad \text{in } L^4(0,\hat{T}; H^1_0(\Omega)) \times C([0,\hat{T}]; L^2(\Omega)). \tag{6.18}$$

The above convergence and the Sobolev embedding theorem imply

$$F(\bar{\phi}_n, \bar{\sigma}_n) \to F(\bar{\phi}, \bar{\sigma}) \quad \text{in } L^2(0,\hat{T}; V^{-3}(\Omega)). \tag{6.19}$$

Let us denote (u_n, ϕ_n, σ_n) and (u, ϕ, σ) the corresponding solutions of problems (6.11), (6.12) associated to $F(\bar{\phi}_n, \bar{\sigma}_n)$ and to $F(\bar{\phi}, \bar{\sigma})$.

Relation (6.19) and Proposition 4.3 yield that

$$u_n \to u \quad \text{in } L^2(0,\hat{T}; V^6(\Omega)) \cap L^\infty(0,\hat{T}; L^2(\Omega)). \tag{6.20}$$

The above result and Proposition 3.5 imply

$$\sigma_n \to \sigma \quad \text{in } C([0,\hat{T}]; L^p(\Omega)) \quad \forall p \in [1,\infty) \tag{6.21}$$

with σ solution of (6.11d).

Moreover, from Proposition 3.6, we deduce that (σ_n) is bounded in

$$C([0,\hat{T}]; C^2(\bar{\Omega})) \cap H^1(0,\hat{T}; C^1(\bar{\Omega})).$$

Therefore, we deduce from Proposition 5.3 that

$$\|\phi_n\|_{L^2(0,\hat{T};H^2(\Omega))} + \|\phi_n\|_{C([0,\hat{T}];H^1_0(\Omega))} + \|\phi_n\|_{H^1(0,\hat{T};L^2(\Omega))} \tag{6.22}$$

is bounded. From these estimates, and using Proposition 5.2, we deduce that

$$\phi_n \to \phi \quad \text{in } L^2(0,\hat{T}; H^1(\Omega)),$$

where ϕ is the weak solution of (6.11c), (6.12). Combining this convergence with (6.22), we obtain that

$$\phi_n \to \phi \quad \text{in } L^4(0,\hat{T}; H^1(\Omega)),$$

which concludes the proof with (6.21). □

7 Proof of the main result

This section is devoted to the proof of Theorem 1.1. First, we take a sequence $\{\sigma_n\}$, with $\sigma_0^n \geqslant 0$ in Ω, $\sigma_0^n \in C^2(\overline{\Omega})$, and

$$\sigma_0^n \to \sigma_0 \quad \text{in} \quad L^p(\Omega) \quad \text{for all } p \in [1,\infty). \tag{7.1}$$

We apply Proposition 6.3 to solve the approximate system (6.1)–(6.2) with initial conditions $(u_0, \phi_0, \sigma_0^n)$ and with a viscosity $\nu = 1/n$. From Proposition 6.2, we deduce that the solution (u_n, ϕ_n, σ_n) verifies

$$\begin{aligned} &\int\limits_\Omega \left(|u_n(t)|^2 + \frac{\lambda}{2}|\nabla\phi_n(t)|^2 + W(\phi_n(t)) + \sigma_n(t)\sqrt{|\nabla\phi_n(t)|^2+\epsilon}\right) \mathrm{dx} \\ &\qquad + \int\limits_0^t \int\limits_\Omega |\nabla u_n|^2 + \left|W'(\phi_n) + A_\sigma(\phi_n)\right|^2 \mathrm{dx}\,\mathrm{d}s + \frac{1}{n}\int\limits_0^t\int\limits_\Omega |\Delta^3 u_n|^2\,\mathrm{dx}\,\mathrm{d}s \\ &\qquad\qquad \leqslant \int\limits_\Omega \left(|u_0|^2 + \frac{\lambda}{2}|\nabla\phi_0|^2 + W(\phi_0) + \sigma_0^n\sqrt{|\nabla\phi_0|^2+\epsilon}\right)\mathrm{dx}, \end{aligned} \tag{7.2}$$

and

$$\sigma_n \geqslant 0 \quad \text{in} \quad Q^T, \quad \|\sigma_n(t)\|_{L^\infty(\Omega)} = \|\sigma_0^n\|_{L^\infty(\Omega)} \leqslant C\|\sigma_0\|_{L^\infty(\Omega)}, \quad t\in(0,T). \tag{7.3}$$

The two above equations imply the following bounds:

$$\{\phi_n\} \quad \text{is bounded in} \quad L^\infty([0,T];H_0^1(\Omega)), \tag{7.4}$$

$$\{u_n\} \quad \text{is bounded in} \quad L^2(0,T;V^1(\Omega)) \cap L^\infty([0,T];V^0(\Omega)), \tag{7.5}$$

$$\left\{\frac{1}{\sqrt{n}}\Delta^3 u_n\right\} \quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega)), \tag{7.6}$$

$$\{W'(\phi_n)\} \quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega)), \tag{7.7}$$

$$\{A_{\sigma_n}(\phi_n)\} \quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega)). \tag{7.8}$$

Note that (7.7) is a consequence of (3.4) and (7.4), whereas, (7.8) is obtained by using (7.7) and (7.2).

Coming back to the equations of ϕ_n and u_n, applying Proposition 3.1 and using the above estimates, we deduce that

$$\{\phi_{n,t}\} \quad \text{is bounded in} \quad L^2(0,T;L^1(\Omega)) \tag{7.9}$$

and

$$\{u_{n,t}\} \quad \text{is bounded in} \quad L^2(0,T;V^{-6}(\Omega)). \tag{7.10}$$

Therefore, from the classical Banach–Alaoglu theorem, Proposition 3.5 and the compactness results given in Subsection 3.2, we finally obtain the following convergences (up to subsequences):

$$\phi_n \rightharpoonup \phi \quad L^\infty(0,T;H_0^1(\Omega)) - \text{weak*}, \tag{7.11a}$$

$$A_{\sigma_n}(\phi_n) \rightharpoonup \chi \quad L^2(0,T;V^0(\Omega)) - \text{weak}, \tag{7.11b}$$

$$\frac{1}{\sqrt{n}}\Delta^3 u_n \rightharpoonup \Phi \quad L^2(0,T;V^0(\Omega)) - \text{weak}, \tag{7.11c}$$

$$u_n \rightharpoonup u \quad L^2(0,T;V(\Omega)) - \text{weak}, \tag{7.11d}$$

and

$$\phi_n \to \phi \quad \text{strongly in} \quad C([0,T];L^4(\Omega)), \tag{7.12a}$$
$$u_n \to u \quad \text{strongly in} \quad L^2(0,T;V^0(\Omega)), \tag{7.12b}$$
$$\sigma_n \to \sigma \quad \text{strongly in} \quad C([0,T];L^p(\Omega)), \quad \forall p \in [1,\infty). \tag{7.12c}$$

Moreover, applying again Proposition 3.5, we deduce that σ is the unique weak solution of transport equation

$$\sigma_t + \nabla\cdot(\sigma u) = 0, \quad \sigma(0,x) = \sigma_0.$$

To pass to the limit in the phase-field equation, we apply Proposition 5.2: we notice that we have the following estimates

$$\frac{1}{2}\int_\Omega |\phi_n(T)|^2\,\mathrm{dx} + \int_0^T \langle A_{\sigma_n}(\phi_n),\phi_n\rangle\,\mathrm{dt} + \int_{Q^T} \phi_n W'(\phi_n)\,\mathrm{dx}\,\mathrm{dt} = \frac{1}{2}\int_\Omega |\phi_n(0)|^2\,\mathrm{dx} \tag{7.13}$$

and

$$\frac{1}{2}\int_\Omega |\phi(T)|^2\,\mathrm{dx} + \int_0^T \langle \chi,\phi\rangle\,\mathrm{dt} + \int_{Q^T} \phi W'(\phi)\,\mathrm{dx}\,\mathrm{dt} = \frac{1}{2}\int_\Omega |\phi(0)|^2\,\mathrm{dx}\,. \tag{7.14}$$

As a consequence, $\chi = A_\sigma(\phi)$ and

$$\nabla\phi_n \to \nabla\phi \quad \text{strongly in} \quad L^2(0,T;L^2(\Omega)).$$

Gathering the above convergence with (7.12c) and with (7.11a), we deduce that

$$F(\phi_n,\sigma_n) \to F(\phi,\sigma) \quad \text{in } L^2(0,T;V^{-3}(\Omega)).$$

This result and relation (7.12b) permit to pass to the limit in the Navier-Stokes system so that u is solution of (1.2a)–(1.2c).

This ends the proof of the theorem.

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